Chapter 2. Declarative Semantics

- Last updated: April 23, 2013 -

How do we know what a goal / program means?

→ Translation of Prolog to logical formulas

How do we know what a logical formula means?

→ Models of logical formulas (Declarative semantics) ← Now

→ Proofs of logical formulas (Operational semantics) ← Later

Question

Question

- What is the meaning of this program?

```
bigger(elephant, horse).
bigger(horse, donkey).
is_bigger(X, Y) :- bigger(X, Y).
is_bigger(X, Y) :- bigger(X, Z), is_bigger(Z, Y).
```

Rephrased question: Two steps

1. How does this program translate to logic formulas?
2. What is the meaning of the logic formulas?

Semantics: Translation

How do we translate a Prolog program to a formula in First Order Logic (FOL)?

→ Translation Scheme

Can any FOL formula be expressed as a Prolog Program?

→ Normalization Steps

Translation of Prolog Programs

1. A Prolog program is translated to a set of formulas, with each clause in the program corresponding to one formula:

   $\{$ bigger(elephant, horse),

   bigger(horse, donkey),

   $\forall x.\forall y.($ bigger(x, y) $\rightarrow$ is_bigger(x, y) $)$,

   $\forall x.\forall y.(\exists z.($bigger(x, z) $\wedge$ is_bigger(z, y)$) \rightarrow$ is_bigger(x, y) $)$

   $\}$

2. Such a set is to be interpreted as the conjunction of all the formulas in the set:

   bigger(elephant, horse) $\wedge$

   bigger(horse, donkey) $\wedge$

   $\forall x.\forall y.($ bigger(x, y) $\rightarrow$ is_bigger(x, y) $)$ $\wedge$

   $\forall x.\forall y.(\exists z.($bigger(x, z) $\wedge$ is_bigger(z, y)$) \rightarrow$ is_bigger(x, y) $)$

Translation of Clauses

- Each comma separating subgoals becomes $\wedge$ (conjunction).
- Each :- becomes $\leftarrow$ (implication)
- Each variable in the head of a clause is bound by a $\forall$ (universal quantifier)
  - `son(X,Y) :- father(Y,X),male(X).`
  - $\forall x.\forall y \; \mathrm{son}(x, y) \leftarrow \mathrm{father}(y, x) \wedge \mathrm{male}(x)$
- Each variable that occurs only in the body of a clause is bound by a $\exists$ (existential quantifier)
  - `grandfather(X):-      father(X,Y) , parent(Y,Z).`
  - $\forall x. \left(\mathrm{grandfather}(x) \leftarrow \exists y.\exists z.\; \mathrm{father}(x, y) \wedge \mathrm{parent}(y, z)\right)$

Translating Disjunction

- Disjunction is the same as two clauses:

```
disjunction(X) :-
    ( ( a(X,Y), b(Y,Z) )
    ; ( c(X,Y), d(Y,Z) )
    ).
```

```
disjunction(X) :-
    a(X,Y), b(Y,Z).
disjunction(X) :-
    c(X,Y), d(Y,Z) .
```

- Variables with the same name in different clauses are different
- Therefore, variables with the same name in different disjunctive branches are different too!
- Good Style: Avoid accidentally equal names in disjoint branches!
  - Rename variables in each branch and use explicit unification

```
disjunction(X) :-
    ( (X=X1, a(X1,Y1), b(Y1,Z1) )
    ; (X=X2, c(X2,Y2), d(Y2,Z2) )
    ).
```

```
disjunction(X1) :-
    a(X1,Y1), b(Y1,Z1).
disjunction(X2) :-
    c(X2,Y2), d(Y2,Z2) .
```

Declarative Semantics – in a nutshell

Meaning of Programs (in a nutshell)

Meaning of a program

Meaning of the equivalent formula.

bigger(elephant, horse)

∧

bigger(horse, donkey)

∧

∀x.∀y.(bigger(x, y) → is_bigger(x, y))

∧

∀x.∀y.(∃z.(bigger(x, z) ∧ is_bigger(z, y)) → is_bigger(x, y))

Meaning of a formula

Set of logical consequences

bigger(elephant, horse)

∧

bigger(horse, donkey)

∧

is_bigger(elephant, horse)

∧

is_bigger(horse, donkey)

∧

is_bigger(elephant, donkey)

Meaning of Programs

Model =
Set of logical consequences =
What is true according to the formula

Meaning of a program

Meaning of the equivalent formula.

bigger(elephant, horse)

$\wedge$

bigger(horse, donkey)

$\wedge$

$\forall x.\forall y.(\ bigger(x, y) \rightarrow is\_bigger(x, y)\ )$

$\wedge$

$\forall x.\forall y.(\ \exists z.(bigger(x, z) \wedge is\_bigger(z, y)) \rightarrow is\_bigger(x, y)\ )$

Meaning of a formula

Set of logical consequences

bigger(elephant, horse)

$\wedge$

bigger(horse, donkey)

$\wedge$

is_bigger(elephant, horse)

$\wedge$

is_bigger(horse, donkey)

$\wedge$

is_bigger(elephant, donkey)

Semantics of Programs and Queries (in a nutshell)

Program

```
bigger(elephant,horse).

bigger(horse,donkey).

is_bigger(X,Y) :-
   bigger(X,Y).

is_bigger(X,Y) :-
   bigger(X,Z),
   is_bigger(Z,Y).
```

Formula

bigger(elephant, horse)
$\wedge$
bigger(horse, donkey)
$\wedge$
$\forall$x.$\forall$y.(is_bigger(x, y) $\leftarrow$
bigger(x, y))
$\wedge$
$\forall$x.$\forall$y.($\exists$z.(is_bigger(x, y) $\leftarrow$
bigger(x, z) $\wedge$
is_bigger(z, y)))

Model

bigger(elephant, horse)
$\wedge$
bigger(horse, donkey)
$\wedge$
is_bigger(elephant, horse)
$\wedge$
is_bigger(horse, donkey)
$\wedge$
is_bigger(elephant, donkey)

Query

?-
bigger(elephant, X)
$\wedge$
is_bigger(X, donkey)

Translation

Interpretation
(logical consequence)

Matching

Declarative Semantics – the details

→ Interpretations of formulas

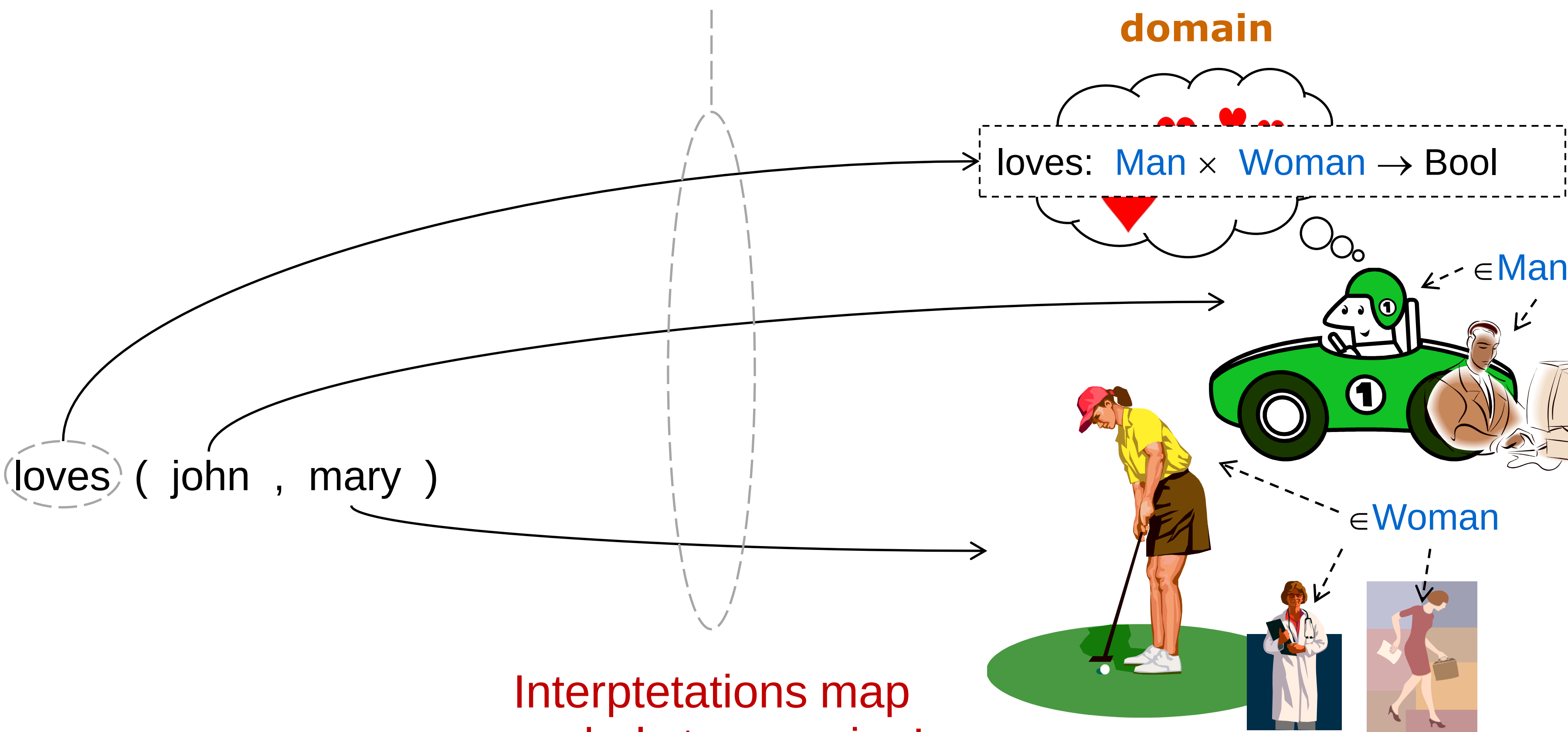
→ Herbrand Interpretations

→ Herbrand Model

→ Logical Consequence

Interpretations of Formulas

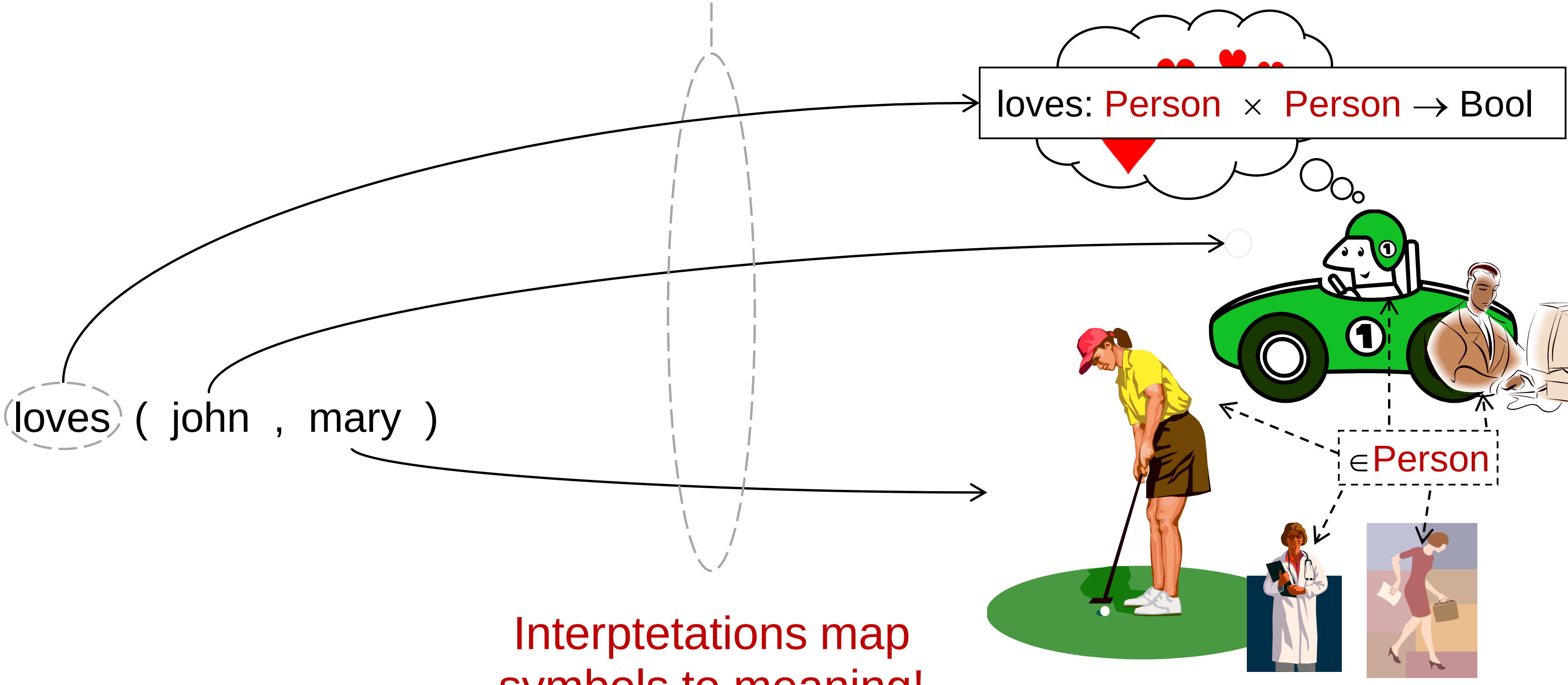
A formula | **An Interpretation** | **An interpretation domain**



Interptetations map symbols to meaning!

Interpretations of Formulas

Same formula | **Slightly different Interpretation** | **Slightly different interpretation domain**



Interptetations map symbols to meaning!

Interpretations of Formulas

Same formula	Other Interpretation	Other interpretation domain

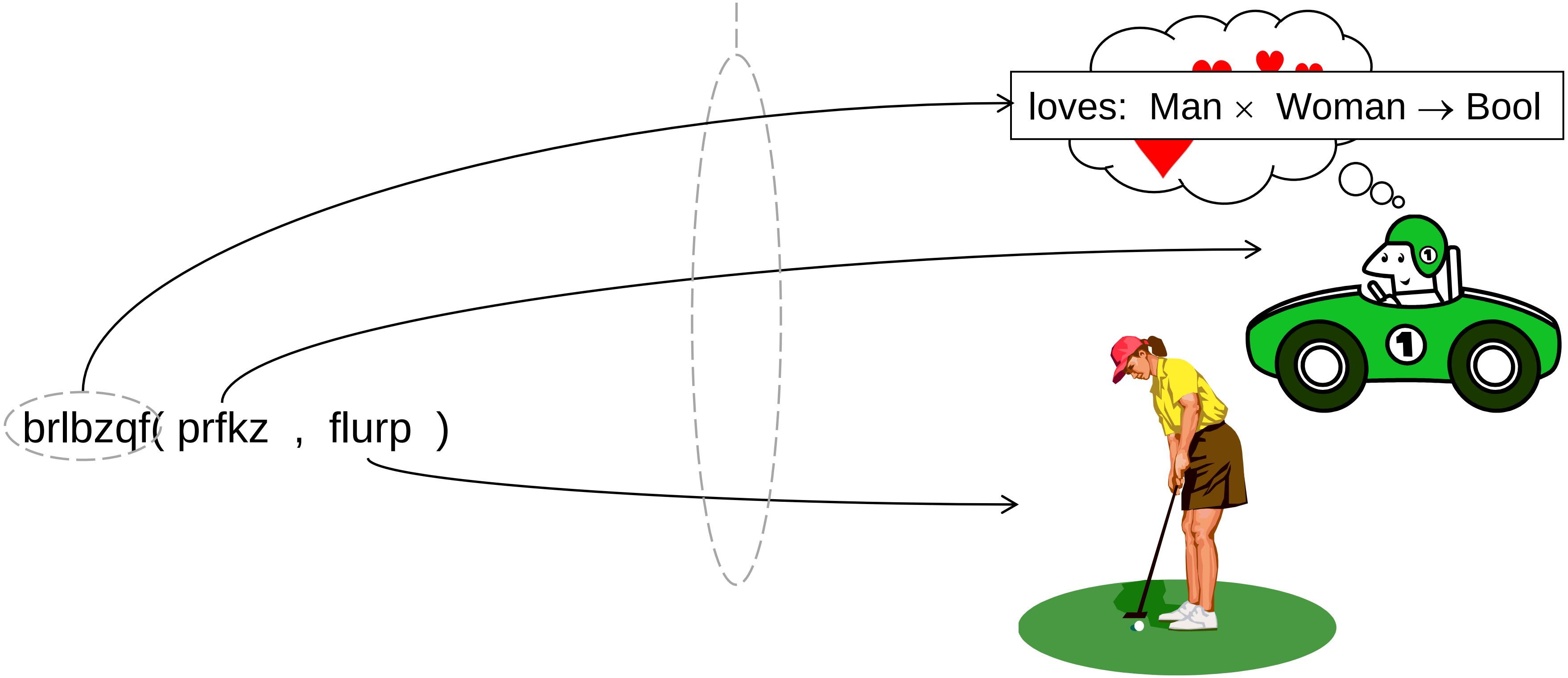
targetOf: Spy × Secret → Bool

loves (john , mary)

Interptetations map symbols to meaning!

Interpretations of Formulas

A formula | **An Interpretation** | **Our initial interpretation domain**



What does this tell us?

Observations

- Formulas only have a meaning with respect to an interpretation
- An interpretation maps formulas to elements of an interpretation domain
  - **constants** to constant in the domain
    - ⇨ e.g. “john” to 
  - **function symbols** to functions on the domain
    - ⇨ function symbols do not occur in our example
  - **predicate symbols** to predicates on the domain
    - ⇨ e.g. “loves/2” to “targetOf: Spy $\times$ Secret $\rightarrow$ Bool”
  - **formulas** to truth values
    - ⇨ e.g. “loves(john,mary)” to “true”

What does this tell us?

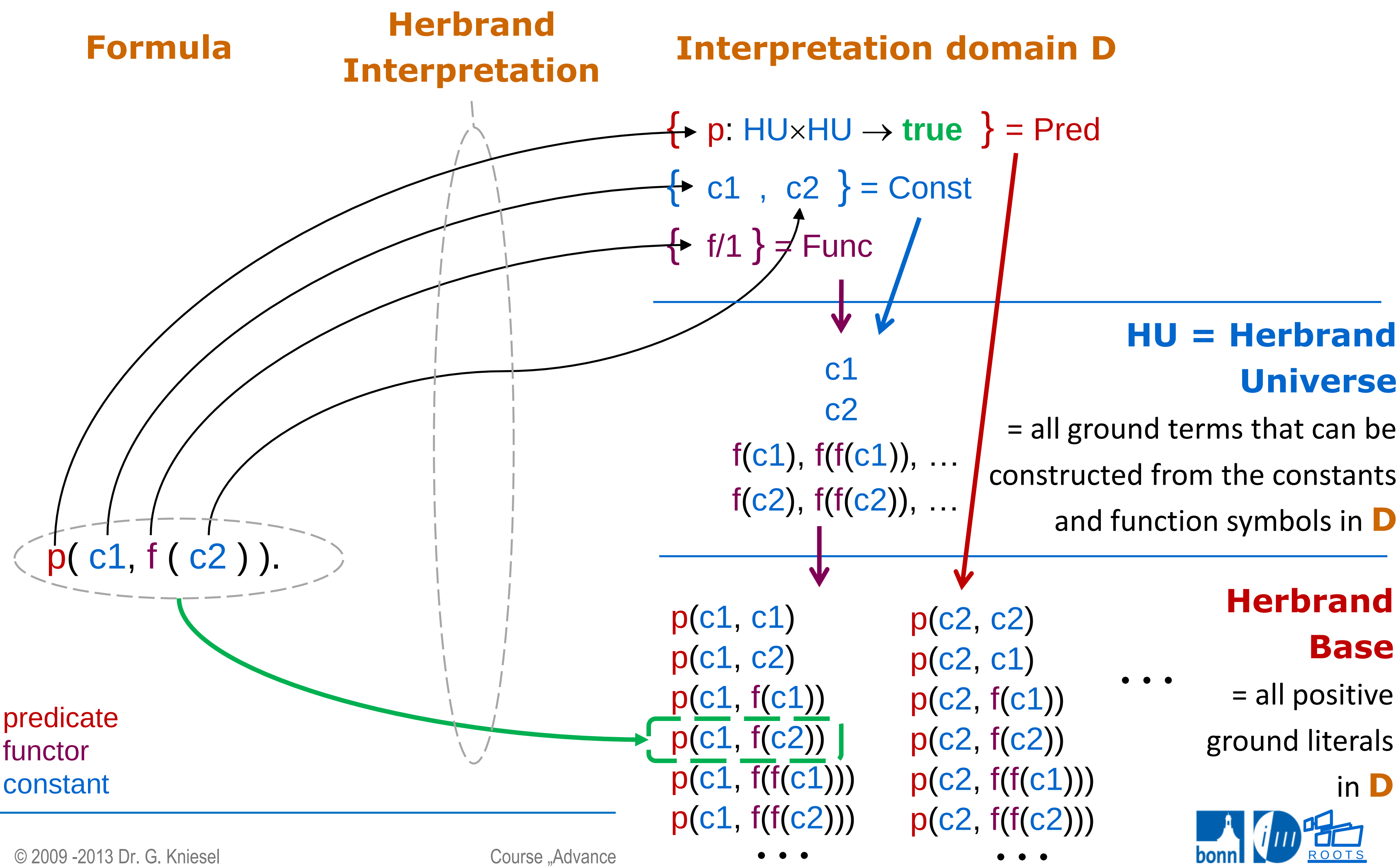
Dilemma

- Too many possible interpretations!
- Which interpretation to use for proving truth?

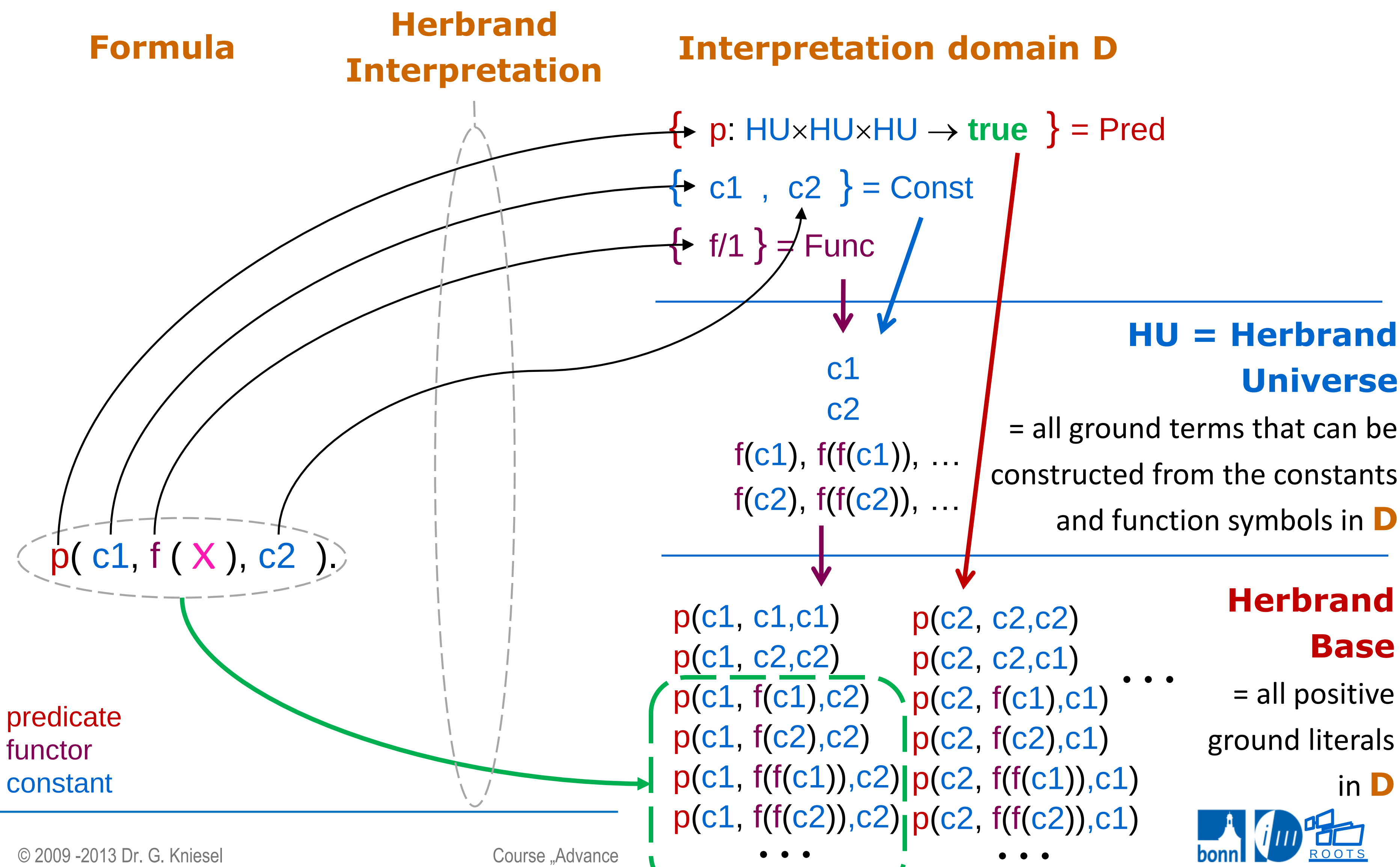
Solution

- For universally quantified formulas there is a “standard” interpretation, the “Herbrand interpretation”, which has two nice properties:
  - If any interpretation satisfies a given set of clauses *S* then there is a Herbrand interpretation that satisfies them
    - ⇨ It suffices to check satisfiability for the Herbrand interpretation!
  - If *S* is unsatisfiable then there is a finite unsatisfiable set of ground instances from the Herbrand base defined by *S*.
    - ⇨ Unsatisfiability can be checked finitely

Herbrand Interpretations



Herbrand Interpretations of Formulas with Variables



Herbrand Models (1)

- The Interpretation Domain (D) of a program P consists of three sets:
  - Const contains all constants occurring in P
  - Func contains all function symbols occurring in P
  - Pred contains a predicate $p\colon \underbrace{HU \times \ldots \times HU}_{\text{arity } n} \rightarrow true$

    for each predicate symbol p of arity n occurring in the program P

- The Herbrand Universe (HU) of a program P is the set of all ground terms that can be constructed from the function symbols and constants in P

= not negated

- The Herbrand Base of a program P is the set of all positive ground literals that can be constructed by applying the predicate symbols in P to arguments from the Herbrand Universe of P

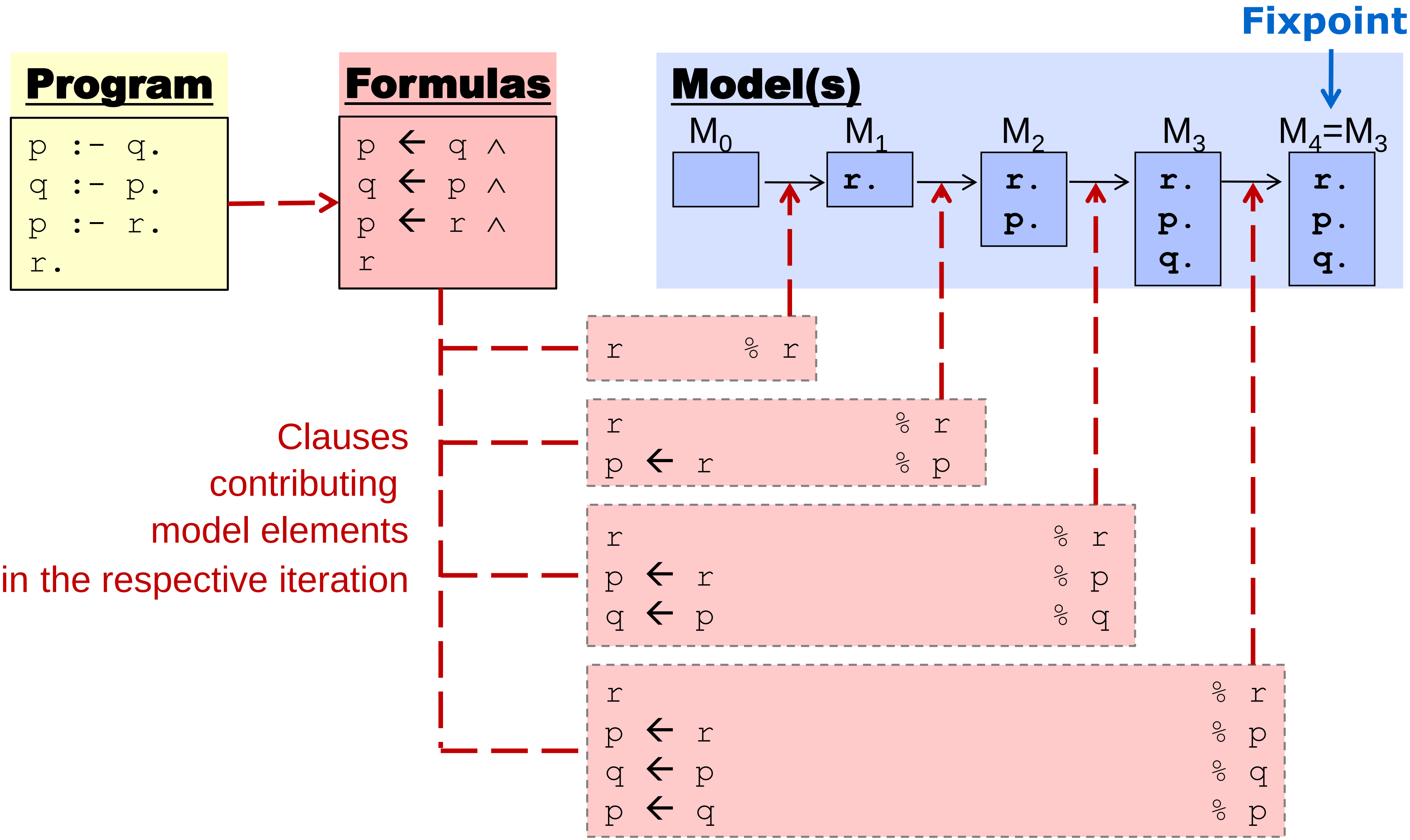
Herbrand Models (2)

- A Herbrand Interpretation maps each formula in a program P to the elements of the Herbrand Base that are its logical consequences
  - Each ground fact is mapped to true.
  - Each ground instantiation of a non-ground fact is mapped to true.
  - Each ground instantiation of the head literal of a rule
    that is a logical consequence of the rule body is mapped to true

- The Herbrand Model of a program P is the subset of the Herbrand Base of P that is true according to the Herbrand Interpretation.
  - It is the set of all logical consequences of the program

- The Herbrand Model of P can be constructed by fixpoint iteration:
  - Initialize the model with the ground instantiations of facts in P
  - Add all new facts that follow from the intermediate model and P
  - … until the model does not change anymore (= fixpoint is reached)

Constructing Models by Fixpoint Iteration



Model-based Semantics → Algorithm

Model-based semantics

- Herbrand interpretations and Herbrand models
- Basic step = “Entailment” (Logical consequence)
- A formula is true if it is a logical consequence of the program

Algorithm = Logic + Control

- Logic = Clauses
- Control =
  - Bottom-up fixpoint iteration to build the model
  - Matching of queries to the model

Program

```
bigger(elephant,horse).
bigger(horse,donkey).
...
```

Formula

bigger(elephant, horse)
$\wedge$
bigger(horse, donkey)
$\wedge$
…

Model

bigger(elephant, horse)
$\wedge$
bigger(horse, donkey)
$\wedge$
…

Query

?-
bigger(elephant, X)
$\wedge$
is_bigger(X, donkey)

Translation

Interpretation
(logical consequence)

Matching

Declarative Semantics Assessed

Pro

- Simple
  - Easy to understand
- Thorough formal foundation
  - implication (entailment)

Perfect for understanding the meaning of a program

Contra

- Inefficient
  - Need to build the whole model in the worst case
- Inapplicable to infinite models
  - Never terminates if the query is not true in the model

Bad as the basis of a practical interpreter implementation

Chapter Summary

- Translation to logic
  - From clauses to formulas
- Declarative / Model-based Semantics
  - Herbrand Universe
  - Herbrand Interpretation
  - Herbrand Model
- Operational interpretation
  - Model construction by fix-point iteration
  - Matching of goals to the model
- Assesssment
  - Strength
  - Weaknesses