
# (Feed Forward) Neural Networks



Dr. Hamed Shariat Yazdi, Prof. Jens Lehmann


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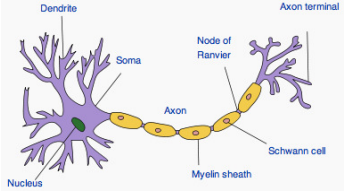
- Today we will learn about a new class of algorithms which can be applied for SRL, namely neural networks.
- We will see how to apply them for SRL in the following lecture.

# Artificial Neurons




source: Wikipedia

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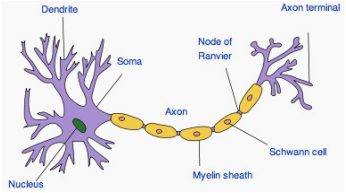


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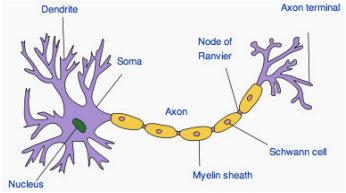


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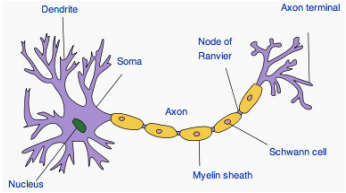


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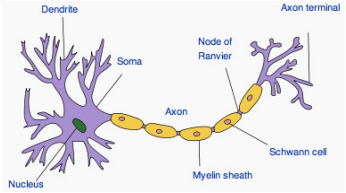


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  Basic idea: reduce the neuron to its essentials.
- There are a number of greatly simplified neuron models. One of the simplest models is given by McCulloch and Pitts.

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- The weight is positive for excitatory and negative for inhibitory connections.
- The absolute value of the weight is small for weak connections and large for strong connections.

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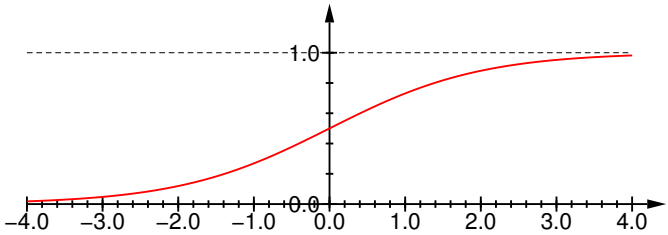

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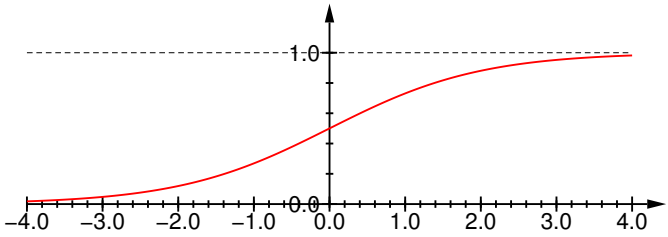


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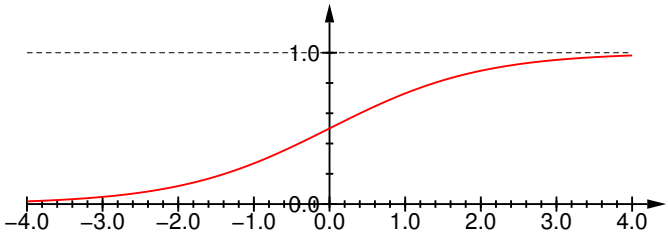


We call such a function a sigmoid or sigmoidal function. The standard formula is:

$$\nu = \sigma(u) = \frac{1}{1 + e^{-u}}$$

# Artificial Neurons

- Now assume we have a pool of neurons, numbered $1, \ldots, n$. Let $u_i$ be the membrane potential and $\nu_i$ be the firing rate of neuron number $i$, and let $w_{ji}$ be the synaptic weight of the connection from $i$ to $j$ (which is zero if the neurons are not connected).

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- This model draws inspiration from biology. However, it is so abstract that in the end it has little in common with its biological counterpart. It should rather be viewed as a computational unit in a mathematical learning machine.

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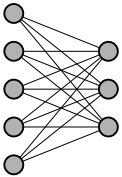
- The perceptron is a model of a single neuron.
- Usually many neurons process the input, so we have multiple perceptrons in parallel:



# Layered Neural Networks

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- Neurons can not only receive input from sensors, but also from other neurons. What if we feed the outputs into another layer of neurons?

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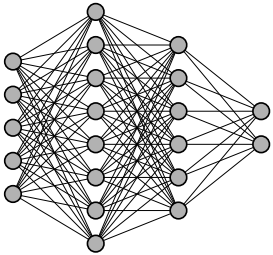
- The resulting architecture is called a (layered) feed-forward neural network, or multi layer perceptron (MLP).



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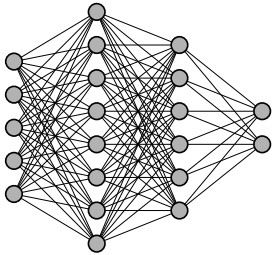

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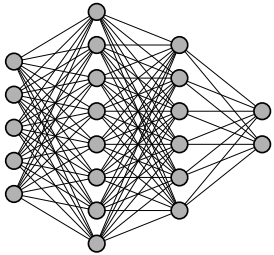

- This example network has an **input layer** with 5 nodes, two **hidden layers** with 8 and 6 neurons, and an **output layer** with 2 neurons.
- The size of the input and output layers is determined by the problem (dimension of the vectors $x$ and $y$), but number and size of the hidden layers is arbitrary.

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▷ Now let $\nu^{(0)}$ denote the vector of inputs, let $\nu^{(i)}$ denote the vector of firing rates in layer $i$, and let $W^{(i)}$ denote the matrix of connections from layer $(i-1)$ to layer $i$. Then we have the overall model:

$$\nu^{(n)} = \sigma_{\text{out}}\left(W^{(n)} \cdot \sigma\left(W^{(n-1)} \cdot \sigma\left(\ldots \sigma\left(W^{(2)} \cdot \sigma\left(W^{(1)} \cdot \nu^{(0)}\right)\right)\ldots\right)\right)\right)$$

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  - Output $\hat{y} = \nu^{(n)}$ (which is hopefully close to the true label $y$).

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- ▷ This is effectively the same as embedding affine functions into linear functions, one dimension up.

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- The non-linearities are not adaptive, but they are nevertheless important! Without them the model would collapse into the linear map $W = W^{(n)} \cdot \ldots \cdot W^{(1)}$. Then all computations were linear.
- It turns out that a neural network with sigmoid transfer functions is indeed far more powerful than a linear model. In a sense, it can compute "everything".

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**Corollary 1.** The theorem extends trivially to multiple outputs.

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[1] If f is a continuous real-valued function on $[a, b]$ and if any $\epsilon > 0$ is given, then there exists a polynomial $p$ on $[a, b]$ such that $|f(x) - P(x)| < \epsilon$ for all $x \in [a, b]$.

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- ▷ **Auto-encoders, (restricted) Boltzmann machines**, and **self-organizing maps** are used for unsupervised learning.

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- A central concept of deep learning is that lower layers extract basic features (e.g. edge detectors in images), while higher layers compose them to complex features (e.g. complex cells, invariant object detectors in images).
- This is a rough correspondence with our understanding of how the visual cortex processes images.
- Recently deep learning revolutionized a lot of fields like image and language procession, machine translations etc. It was also part of Alpha-Go (the system developed by Google to play the board game "Go").

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- ▷ With a large enough hidden layer the network can approximate the optimal hypothesis arbitrarily well.
- ▷ However, this is not helpful in practice. It does not tell us how to actually set the network size, let alone the weights.
- ▷ Until now we have defined neural networks as a class of models. We do not have a learning rule yet.
- ▷ Neural networks are trained based on **stochastic gradient descent** as described in the following.

# (Online) Steepest Descent Training

- Let $w$ denote a vector collecting all weights of a neural network. This is a "linearized" version of all of its weight matrices.

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$$E(w) = \frac{1}{|S|} \sum_{i \in S} L(f_w(x_i), y_i)$$

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  denote the error of the network, as a function of the weights.
- The sum may run over the whole data set ($|S| = n$, batch mode), over small subsets ($|S| \ll n$, mini batches), or only over a single example ($|S| = 1$, online mode).

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- This is usually what we want when training a neural network. So why care about online and mini batch errors?
- The reason is that the online error is much faster to compute, namely by a factor of $n$ (size of the data set). Thus its use allows for many more gradient descent steps.

# (Online) Steepest Descent Training

- Assume we have computed the gradient of the error $\nabla_w E(w)$ with respect to the weights. Then we can perform a step of gradient descent with learning rate $\eta$ to update the wights.

$$w \leftarrow w - \eta \cdot \nabla_w E(w) \ .$$

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  where $W^{(k)}$ are the weight matrices and $\sigma$ is the component-wise non-linearity.
- The gradient can be calculated by the chain rule.
- **Backpropagation** is an algorithm for doing this fast. It will be introduced in the next lecture.

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- Neural networks are universal function approximators.
- Networks can be trained by online or batch gradient descent.
- The error gradient can be computed efficiently with the backpropagation algorithm.
- The weights usually end up in a local optimum, not in the global optimum.

# Acknowledgments

Acknowledgments: We thank Tobias Glasmachers for providing us the material for this class which was taken from his lecture *Machine Learning - Supervised Methods* at the Ruhr-Uni-Bochum.